Harvesting economic models and catch-to-biomass dependence: The case of small pelagic fish

Pedro Gajardo* Julio Peña-Torres† Héctor Ramírez C.‡

Abstract

In the case of small pelagic fish it seems reasonable to consider harvest functions depending nonlinearly on fishing effort and fish stock. Indeed, empirical evidence about these fish species suggests that marginal catch does not necessarily react in a linear way neither to changes in fishing effort nor in fish stock levels. This is in contradiction with traditional fishery economic models where catch-to-input marginal productivities are normally assumed to be constant. While allowing for non linearities in both catch-to-effort and catch-to-stock parameters, this paper extends the traditional single-stock harvesting economic model by focusing on the dependence of the stationary solutions upon the nonlinear catch-to-stock parameter. Thus, we analyze equilibrium responses to changes in this parameter, which in turn may be triggered either by climatic or technological change. Given the focus in this study on the case of small pelagic fish, the analysis considers positive but small values for the catch-to-stock parameter.

Keywords: small pelagic fisheries, Cobb-Douglas production function, optimal control, maximum principle.

1 Introduction

Small pelagic fish stocks, such as anchovy, sardine, herring and jack mackerel represent an important proportion of world's marine fish harvests (currently about a third of it, with more than 20 million tons per year worldwide, according to FAO statistics; (Herrick, Norton, Hannesson, Sumaila, Ahmed, and Peña-Torres 2009)). In some fishing nations (e.g., Peru and Chile), fisheries of this type are important resources for their national economies, both in terms of value added production as well as for regional employment.

*Departamento de Matemática, Universidad Técnica Federico Santa María, Avda. España 1680 Casilla 110-V, Valparaíso, Chile. Email: pedro.gajardo@usm.cl

†Facultad de Economía y Negocios/ILADES, Universidad Alberto Hurtado. Erasmo Escala 1835, Santiago, Chile. Email: jpena@uahurtado.cl

‡Departamento de Ingeniería Matemática, Centro de Modelamiento Matemático (CNRS UMI 2807), FCFM, Universidad de Chile, Avda. Blanco Encalada 2120, Santiago, Chile. Email: hramirez@dim.uchile.cl

Small pelagic stocks are characterized by some peculiar features. On the one hand, they tend to face strong and recurrent cycles of fish abundance. On the other, they usually provide high catch yields per unit of fishing effort, even if the stock is being depleted, i.e. the catch per unit of fishing effort can be fairly independent of the stock size. Given these characteristics, different pelagic stocks have experienced fishing collapse. Examples in the XXth century are the sardine fishery in Japan during the early 1940s, the sardine fishery in California a decade later, the herring population in the North Sea and the Atlanto-Scandian herring stock at the end of the 1960s and early 1970s, and the early 1970s collapse of the Peruvian anchovy (Peña-Torres 1996).

The above-mentioned characteristics about small pelagic fish are normally absent from traditional mathematical fishery economic models. This literature has typically focused on the case of linear harvest functions (well known examples are (Clark 1980; Clark 1990); (Plourde and Yeung 1987); (Dockner, Feichtinger, and Mehlmann 1989)). This approach has obvious advantages in terms of mathematical tractability. Thus, the model that usually describes a single species fish stock's evolution in continuous time is

$$\dot{x}(t) = F(x(t)) - u(t)x(t),$$

where $x(t)$ is the fish stock level at time t, $u(t)$ is fishing effort[1] and F is the species' biological growth function.

In contrast, and given this paper's focus on the case of small pelagic fish resources, we consider a Cobb-Douglas form for the harvest function[2]:

$$h(t) = u^{\alpha}(t)x^{\beta}(t), \tag{1.1}$$

where $h(t)$ is the rate of harvesting at time t (measured in tons per unit of time), and α and β are two positive parameters such that $\alpha + \beta \geq 1$.

The value of parameter α controls for how fishing effort's marginal catch productivity (yields) varies as fishing effort changes. Thus, $\alpha = 1$ implies constant marginal catch productivity of additional fishing effort units. On the other hand, the parameter β measures how sensitive catch yields are to marginal changes in fish stock level. In the case of constant unit cost of fishing effort, the lower the value of β the less sensitive the unit harvest cost will be to variations in fish stock level. Hence, the lower the value of β the more likely should be, *ceteris paribus*, the occurrence of fishing collapse.

In relation to literature about applied fisheries stock assessment methods, the case $\alpha = \beta = 1$ describes a fishery context where the catch per unit of fishing effort (CPUE) is proportional, in a time-invariant fashion, to stock abundance. In such a case, fishery-dependent data such as the CPUE could be used, in principle, as an index for estimating

[1]The use of a single input variable presupposes that other inputs (e.g., labour, capital) are used in fixed proportions, so input use intensity can be measured up by a single variable.

[2]This is a widely used functional form in economics. See (Heathfield and Wibe 1987, Chapter 4) about its properties when applied to modeling production functions. However, its use at fishery models has been very uncommon. Rare exceptions are (Léonard and Long 1992), (Hannesson 1993, Page 53), (Hannesson and Kennedy 2005) and (Peña-Torres 1995). See also (Dasgupta and Heal 1979) for a classical description of its use at optimal economic growth models for economies with exhaustible natural resource.

unknown fish abundance. However, there are different reasons why the relationship between CPUE and stock abundance could be both non-linear and time-varying. Regarding the case of schooling fish species, such as small pelagics, different empirical studies have indeed pinpointed the relevance of a non-linear relationship between CPUE and stock abundance (for more details, see (Wilberg et al. 2010); (Quinn and Deriso 1999, Chapter 1); (Hilborn and Walters 1992);(Csirke 1988)).

In effect, and regarding function (1.1), different empirical studies about small pelagic fisheries suggest that the conjecture "α and β would simultaneously be close to one" should be rejected. Indeed, available evidence suggests positive values but lower than the unit for the case of β (e.g., (Opsomer and Conrad 1994); (Bjorndal and Conrad 1987)). For this type of fish stocks, some authors have even suggested that, for certain ranges of fish abundance, total independence may eventually prevail between catch yields and fish stock levels (e.g., (MacCall 1976); (Clark 1982); (Csirke 1988); (Bjorndal 1988; Bjorndal 1989))[3]. Regarding the value of α, available evidence for several small pelagic fisheries suggests positive values that are either close to or even greater than one (e.g. (Bjorndal 1987; Bjorndal 1989); (Bjorndal and Conrad 1987); (Opsomer and Conrad 1994); (Peña-Torres and Basch 2000); (Peña-Torres, Vergara, and Basch 2004)). Thus, in order to maintain mathematical tractability, in this paper we will limit the analysis to studying cases with $\alpha \leq 1$.

The evidence cited on small pelagic fish confirms the relevance of analyzing fishery contexts with β positive but small enough. Thus, special emphasis should be put on the effects of relatively stock-insensitive, or even declining, average (unit) harvesting costs at reduced stock levels (Peña-Torres 1996). The possibility of stock-insensitive average (unit) harvesting costs arises as the result of defensive strategies that these fish species follow at reduced stock levels: as the stock x declines, fish shoals tend to increase their density as a defense response to natural predators. For species that live near to the sea surface, like small pelagics, the latter effect tends to reduce average (unit) harvesting costs[4].

Therefore, this paper will assume harvesting settings in which $\alpha + \beta = 1$ and β is positive but near to zero. Indeed, our analysis will be focused on the resulting stationary equilibria when β is tending to zero. In particular, given $\alpha + \beta = 1$, that we assume for the sake of mathematical simplicity, we study the effects of changes in the proportion (α/β) when $\beta \to 0$ and $\alpha \to 1$.

To study the sensitivity of a fishery equilibrium to changes in parameter β can provide useful insights not only from the viewpoint of analyzing the risk of fishing collapse, but also for other relevant issues. For example, it may help to anticipate new equilibrium responses

[3]Marine biologists (e.g. (MacCall 1990; Csirke 1988)) have stated that in small pelagic fisheries mean harvest yields (per unit of fishing effort) are not a good predictor of changes in fish abundance. The hypothesis is that when abundance falls, small pelagic stocks tend to reduce the range of their feeding and breeding areas, with concurrent decreases in the number of schools, despite that schools' average size may remain constant. That is, the stock reduces the range of its spatial distribution while simultaneously increasing its density. This behaviour could result in a relation of (at least transitory) independence between harvest yields and fish stock abundance.

[4](Wilberg et al. 2010) is an excellent recent review on different sources of density-dependent changes in fish catchability and thus, more generally, on reasons for getting a non-linear relationship between CPUE and stock abundance.

in a given type of fishery – for which there will be a value of β that summarizes the combined effect of environmental and technological factors upon the interaction between fish abundance and the resulting catch yields per unit of fishing effort–, when the fishery faces changes in the value of its parameter β. The latter may occur as a result of climatic change and its triggered effects on the level, density or location of the fish stock in exploitation, or due to technological or institutional innovations that affect fishermen's capacity to search and catch fish shoals.

The analysis of a fishery's equilibrium response to changes in β can also provide useful insights when the fisheries regulator needs to evaluate the comparative advantage of using landing fees or harvest quotas in a given type of fishery. Indeed, as long as this decision is affected by different types of uncertainty (e.g., about fish location, stock estimates or even economic parameters), the regulator's optimal policy choice will normally depend not only on the type of uncertainty faced but also on the value of parameter β, or of changes in it, as the latter will directly affect the unit cost of harvesting (see (Jensen and Vestergaard 2003; Weitzman 2002; Hannesson and Kennedy 2005)).

For simplicity, we consider the optimal control problem of a fictitious (price-taking) sole owner, say a social planner, who maximizes the total discounted value of the intertemporal flows of the fish stock economic rents, when the harvest function of each agent is given by (1.1). We analyze the steady states of the associated dynamical system describing the asymptotic behavior of these states when β in (1.1) tends to zero.

The outline of this article is as follows: Section 2 introduces the social planner's problem when $\beta \in (0, 1)$ and $\alpha + \beta = 1$. Section 3 analyzes the unique steady state equilibrium's behavior as a function of parameter β. Section 4 discusses the asymptotic behavior of the unique stationary equilibrium when $\beta \to 0$. This section constitutes the main core of our work. Finally, in Section 5 we present our final remarks. Mathematical proofs are relegated to the Appendix.

2 The social planner's problem

We will model in a highly stylized fashion the real-world fishery management problem by assuming that it simply consists of choosing a single decision variable, that is, the fishing effort level. We will also assume, in contrast with real-world fisheries stock assessment practices, that the fish stock under exploitation is perfectly known at any time. We use these simplifying modeling devises so that to focus more directly on analyzing the sensitivity of the social planner's decision about the optimal scale (stationary level) of fishing effort, and hence of the stationary equilibrium levels for the harvest rate and the fish stock, with respect to changes in the parameter β. Recall that this parameter is a modeling artifact which aggregates and summarizes the effect of different possible sources of non-linearity between the CPUE and fish abundance.

Consider N symmetric fishing units (say vessels) harvesting simultaneously a single-species fish stock. The number of fishing units is exogenously defined (say by exogenous political considerations). All these vessels are under the social planner's control.

Given an admissible fishing effort policy $u(\cdot)$, the resulting rate of harvesting $h(\cdot)$ is given by the Cobb-Douglas function (1.1). Thus, the evolution of the fish stock level $x(\cdot)$, starting from an initial condition $x_0 > 0$, is given by the solution of the following ordinary differential equation:

$$\begin{cases} \dot{x}(t) = F(x(t)) - Nu^{\alpha}(t)x^{\beta}(t) & t > 0 \\ x(0) = x_0 \end{cases} \tag{2.1}$$

where the biological growth function F is assumed strictly concave and twice continuously differentiable. We shall assume that there exists $K > 0$ called *saturation constant* such that $F(0) = F(K) = 0$ and $F(x) > 0$ for all $x \in (0, K)$.

Notice that the specification at equation (2.1) assumes the possibility of decreasing harvest returns from marginal fishing effort units only at the vessel level. Thus, aggregate fleet's congestion effects are ruled out by assumption.

Example 2.1. Some examples of biological growth function F are the following:

- Logistic function: $F(x) = ax(1 - x/K)$;
- Gompertz function: $F(x) = \begin{cases} ax \ln(K/x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

where $a > 0$ is a given parameter.

In what follows we assume that $x_0 \in (0, K)$ and therefore the trajectory $x(t)$ remains in this interval for all $t > 0$ and for any applied fishing effort.

The social planner's problem consists of choosing each vessel's fishing effort $u(t) \geq 0$ in order to maximize the total discounted value of the intertemporal flow of the natural resource's rents given by

$$J(u, x) := N \int_0^{+\infty} e^{-rt}(pu^{\alpha}(t)x^{\beta}(t) - cu(t))dt \tag{2.2}$$

where $r > 0$ is the (time invariant) discount rate, c is the (constant) unit cost of fishing effort, and p is the (constant) unit price of harvesting. In order to ensure the positivity of the instantaneous profits above, we will assume from now on that $p > c$.

Hence, for a given initial condition x_0, the infinite horizon control problem is established as follows:

$$(P_{SP}) \quad V(x_0) := \max_{u \in \mathcal{U}} \{J(u, x) \ : \ x \ \text{ solves (2.1)}\} \tag{2.3}$$

where $J(u, x)$ is the criterion given in (2.2) and the admissible control set $\mathcal{U}$ is defined by

$$\mathcal{U} = \{u : [0, +\infty) \longrightarrow [0, +\infty) \ : \ u \text{ piecewise continuous}\}.$$

In what follows we suppose that the maximal value is so high that, in practice, the fishing effort u never reaches it. This explains the range set of the form $[0, +\infty)$ for admissible efforts u in the definition of $\mathcal{U}$.

We will focus on the problem (P_{SP}) when the marginal catch productivity is strictly decreasing with respect to the stock level, that is when $\beta \in (0,1)$. Moreover, for the sake of simplicity, we shall assume $\alpha + \beta = 1$. The latter allows us to work with a strictly concave Hamiltonian.

From the optimal control theory, the Pontryagin's maximum principle establishes (e.g. (Pontryagin, Boltyanskii, Gamkrelidze, and Mishchenko 1962; Bardi and Capuzzo-Dolcetta 1997; Vinter 2000)) that if $u : [0, +\infty) \longrightarrow [0, +\infty)$ is an optimal solution of the infinite horizon problem (P_{SP}) and $x : [0, +\infty) \longrightarrow (0, K)$ the associated fish stock level, then there exists a function λ differentiable almost everywhere (a.e.) such that

$$\dot{\lambda}(t) = r\lambda(t) - \beta N p u^{\alpha} x^{\beta-1} - \lambda(t)(F'(\bar{x}) - \beta N u^{\alpha} x^{\beta-1}) \qquad \text{a.e. } t > 0; \tag{2.4}$$

and, the Hamiltonian defined by

$$H(\lambda, x, u) = N(pu^{\alpha}x^{\beta} - cu) + \lambda(F(x) - Nu^{\alpha}x^{\beta}) \tag{2.5}$$

is maximized in $u(t)$ for every t, that is

$$H(\lambda(t), x(t), u(t)) = \max_{u \geq 0} H(\lambda(t), x(t), u). \tag{2.6}$$

In this formulation, the function λ represents the current valued shadow (unit) price of x.

Therefore, given an optimal policy u and the associated trajectory x, equality (2.6) allows to obtain the expression of u in terms of the shadow price λ and the stock level x. Indeed, since the Hamiltonian H is maximized in $u(t)$, Fermat's rule $\frac{\partial H}{\partial u} = 0$ gives

$$u(x(t), \lambda(t)) = \begin{cases} 0 & \text{if } \lambda(t) \geq p \\ \left(\frac{\alpha(p-\lambda(t))}{c}\right)^{\frac{1}{\beta}} x(t) & \text{if } \lambda(t) < p. \end{cases} \tag{2.7}$$

As expected, the above expression shows that if the social planner's shadow value of keeping an additional unit of fish stock at sea is higher than the harvest price p, then the optimal policy consists in stopping fishing effort completely and immediately.

From (2.1), (2.4), and (2.7) we obtain a new system for the state x and adjoint state λ given by

$$\begin{cases} \dot{x}(t) = \varphi_1(x(t), \lambda(t)) \\ \dot{\lambda}(t) = \varphi_2(x(t), \lambda(t)); \\ x(0) = x_0, \end{cases} \tag{2.8}$$

where

$$\begin{aligned} \varphi_1(x, \lambda) \;&:=\; \begin{cases} F(x) & \text{if } \lambda \geq p \\ F(x) - N\phi^{\alpha}(\lambda)x & \text{if } \lambda < p, \end{cases} \\ \varphi_2(x, \lambda) \;&:=\; \begin{cases} \lambda(r - F'(x)) & \text{if } \lambda \geq p \\ \lambda(r - F'(x)) - \beta N \phi^{\alpha}(\lambda)(p - \lambda) & \text{if } \lambda < p, \end{cases} \end{aligned}$$

and

$$\phi(\lambda) := \left(\frac{\alpha(p-\lambda)}{c}\right)^{\frac{1}{\beta}}.$$

Notice that the functions φ_1 and φ_2 are continuously differentiable. This implies the existence and uniqueness of (x, λ) solution of system (2.8).

3 Properties of the stationary equilibrium

The proposition below ensures the existence and uniqueness of a steady state of the system (2.8).

Proposition 3.1. *If $r < F'(0) < N((1-\beta)p/c)^{\frac{1-\beta}{\beta}}$, then the system* (2.8) *has only one steady state $(x^*(\beta), \lambda^*(\beta))$ satisfying the relation:*

$$\lambda^*(\beta) = p - \left(\frac{c}{1-\beta}\right)\left(\frac{F(x^*(\beta))}{Nx^*(\beta)}\right)^{\frac{\beta}{1-\beta}}. \tag{3.1}$$

Furthermore, the unique steady state is in $(x_r, K) \times (\bar{\lambda}_\beta, p)$, where

$$F'(x_r) = r \quad \text{and} \quad \bar{\lambda}_\beta = p - \frac{c}{(1-\beta)}\left(\frac{F(x_r)}{Nx_r}\right)^{\frac{\beta}{1-\beta}}.$$

Proof. See Appendix A.1. □

Remark 3.2. Notice that the term $N((1-\beta)p/c)^{\frac{1-\beta}{\beta}}$ converges to $+\infty$ when $\beta \to 0$. Then, for small values of β, the hypothesis $F'(0) < N((1-\beta)p/c)^{\frac{1-\beta}{\beta}}$ holds true.

Remark 3.3. The stationary solution $\lambda^*(\beta)$ that is relevant to our analysis is necessarily positive, as the latter is the only solution of economic interest. Notice that the positivity of $\lambda^*(\beta)$ will hold for $\beta > 0$ small enough.

Naturally, we need to impose condition $F'(0) > r$ to ensure that the stationary solution $x^*(\beta)$ will be strictly positive. Otherwise, it would be optimal to fully deplete the resource x and thereby being able to invest the obtained harvesting profits at the market return $r > 0$.

Proposition 3.1 also states that the optimal stationary state $x^*(\beta)$ will be strictly above the value x_r. The logic for this is as follows. First of all, the stationary economic optimum implies that no additional gains can be obtained from exploiting x at a different level. Therefore, at the stationary equilibrium, the return obtainable from marginal investment in x must fully coincide with the opportunity cost of that investment. In our problem (P_{SP}), such opportunity cost is given by the parameter $r > 0$. Whereas the marginal return from investment in x can be obtained by applying the Euler-Lagrange equations to our problem

(P_{SP}). The resulting marginal return is described by the left-hand-side of the following equation:

$$F'(x^*(\beta)) + \frac{Nc\beta}{1-\beta}\frac{\left(\frac{F(x^*(\beta))}{Nx^*(\beta)}\right)^{\frac{1}{1-\beta}}}{\lambda^*(\beta)} = r. \tag{3.2}$$

The latter is a variant (for $0 < \beta < 1$) of the well-known equation describing the stationary optimal solution (x^*, λ^*), often described as the "*fundamental equation of renewable resource exploitation*" (e.g., see (Bjorndal and Munro 1998), and (Hannesson 1993, p. 35)).

By resorting to equation (3.2), we can now explain why the optimal state $x^*(\beta)$ (for $\beta > 0$) will be strictly above x_r.

As (3.2) shows, the return from keeping an additional unit of x at sea comes from two sources. On the one hand, the biological return of keeping an additional unit of x at sea, which is given by $F'(x^*(\beta))$. On the other, the profits resulting from the incremental harvest, given by $\frac{\partial(u^\alpha x^\beta)}{\partial x} > 0$, is positive for $\beta > 0$. This second source of return will thus increase the profitability of investing in x, adding itself to the gain directly consisting of the marginal biological return $F'(x^*(\beta))$. Therefore, the intertemporal equilibrium (that is, the optimal investment) condition will be such that $F'(x^*(\beta)) < F'(x_r) = r$ and, by the strict concavity of function $F(\cdot)$, then $x^*(\beta) > x_r$.

Remark 3.4. It is straightforward to prove that the unique steady state of the system (2.8) is a saddle point. This property arises in many different economic contexts. In the case of harvesting fishery models, it can be found in (Léonard and Long 1992, p. 295) and (Peña-Torres 1995), both considering a logistic function F and $\alpha = \beta = 1/2$.

4 Analysis about the asymptotic behavior when $\beta \to 0$

In this part, we study the behavior of the unique steady state (given by Proposition 3.1) with respect to variations of parameter β. In particular, we focus on the case of $\beta \to 0$. In addition to the relevance of this case for small pelagic fish stocks, its analysis provides general insights about the equilibrium responses to be expected in a 'low β' fishery, when that fishery's β may be changing due to environmental or technological exogenous shocks. This asymptotic analysis also provides insights about the equilibrium responses underlying a fishery that is approaching the well-known case of $\alpha = 1$ and $\beta = 0$. See e.g. (Hannesson 1993, Section 2.8) and (Clark 1990, Section 3.5).

For the parameter $\beta \in (0, 1)$, we shall denote by $x^*(\beta)$ and $\lambda^*(\beta)$ the corresponding (unique) steady states. The pair $(x^*(\beta), \lambda^*(\beta))$ solves the following system of equations

$$\begin{cases} 0 = F(x) - N\phi^{1-\beta}(\lambda)x \\ \\ 0 = -\lambda(F'(x) - r) + \beta N\phi^{1-\beta}(\lambda)(\lambda - p). \end{cases} \tag{4.1}$$

We write $u(\beta)$ for the associated steady control, which is given by

$$u(\beta) = \left(\frac{\alpha(p - \lambda^*(\beta))}{c}\right)^{\frac{1}{\beta}} x^*(\beta). \tag{4.2}$$

Thus, the rate of harvesting at the equilibrium is

$$h(\beta) = Nu(\beta)^\alpha x^*(\beta)^\beta = F(x^*(\beta)). \tag{4.3}$$

The following proposition establishes the continuous dependence of the obtained steady states with respect to parameter β, concluding that the equilibrium points are a differentiable function of this parameter. Additionally, the result below provides the limits of the steady state functions when β goes to zero.

Since our analysis is focused in small β configurations, in the following we shall assume that the assumptions of Proposition 3.1 hold true, that is, $r < F'(0) < N((1-\beta)p/c)^{\frac{1-\beta}{\beta}}$, and therefore for each considered β there exists only one equilibrium point.

Proposition 4.1. *There exist two continuously differentiable functions $x(\cdot)$ and $\lambda(\cdot)$ such that for each $\beta > 0$ small enough, $(x(\beta), \lambda(\beta))$ is the unique solution of the system* (4.1). *Moreover, when the parameter β goes to zero, it holds that:*

1. $\lim_{\beta \to 0} x(\beta) = x_r$;
2. $\lim_{\beta \to 0} \lambda(\beta) = p - c$,

where x_r is such that $F'(x_r) = r$. Finally, the limit of the optimal effort at the equilibrium is

$$\lim_{\beta \to 0} u(\beta) = \frac{F(x_r)}{N}.$$

Proof. See Appendix A.2. □

Remark 4.2. Note that, since the steady state of system (2.8) is unique, Proposition 4.1 above implies that $(x(\beta), \lambda(\beta)) = (x^*(\beta), \lambda^*(\beta))$, for all $\beta \in (0, 1)$. From now on, we can use the notation $(x(\beta), \lambda(\beta))$ for the steady states of (2.8) without any possibility of confusion.

Remark 4.3. It is well known that the solution of the Pontryagin system (2.8) associated with problem (P_{SP}) when $\alpha = 1$ and $\beta = 0$ is a turnpike solution approaching as fast as possible to the values $x = x_r$ and $\lambda = p - c$ (see (Clark 1990)). So, Proposition 4.1 establishes that the limit behavior of the steady states solutions $(x(\beta), \lambda(\beta))$, when $\beta \to 0$, is coherent with this limit result.

When $\beta \to 0$ we see from condition (3.2) that there will tend to remain a unique source of return from keeping an additional unit of x at sea, that is the biological growth rate $F'(x(\beta))$. This is so because, as $\beta \to 0$, the current period profits tend to be independent of x and therefore the Hamiltonian (or value) function at (2.5), which has to be maximized by choosing the optimal control $u(\beta)$, varies with changes in x only by the differential effect $F'(x)$. As a result of this, the optimal stationary equilibrium $x(\beta)$ tends to x_r.

Proposition 4.4. *The limits of the derivatives of steady states with respect to β, when $\beta \to 0$, are:*

$$\lim_{\beta\to 0} \frac{dx}{d\beta} = -\frac{cF(x_r)}{F''(x_r)(p-c)x_r} > 0; \tag{4.4}$$

$$\lim_{\beta\to 0} \frac{d\lambda}{d\beta} = -c\left[\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1\right]. \tag{4.5}$$

Therefore, for β small enough, we have:

1. $\frac{dx}{d\beta} > 0$*, that is, $x(\beta)$ decreases when β decreases.*
2. *(a) if* $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 > 0$ *then* $\frac{d\lambda}{d\beta} < 0$*, that is, $\lambda(\beta)$ increases when β decreases.*

   (b) if $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 < 0$ *then* $\frac{d\lambda}{d\beta} > 0$*, that is, $\lambda(\beta)$ decreases when β decreases.*

Proof. See Appendix A.3. □

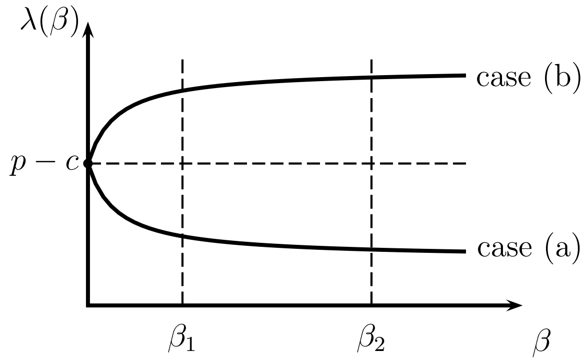

Figure 1: Different behaviors of the steady state $\lambda(\beta)$, depending of configurations described in (a) and (b).

The result in Part 1 of Proposition 4.4 is directly related to the economic intuition already analyzed regarding the results in Proposition 3.1: a greater value of $\beta > 0$ increases the profits from one of the two sources of positive marginal returns that are obtained by keeping an additional unit of x at sea. Thus, at the steady state equilibrium a greater value of β will necessarily imply a higher stationary value for x.

In Part 2 of Proposition 4.4, the result describes, for a given function $F(\cdot)$ and given values of N and r, how the asymptotic stationary solution for $\lambda(\beta)$ is approached when $\beta \to 0$: that is, either approaching it from an initial stationary λ value that is above the stationary solution $p-c$ (case b; see Figure 1), or from a λ value that is below the stationary solution $p-c$ (case a).

The result (2.a) implies that $\frac{d\lambda}{d\beta} < 0$ if the value[5] $F(x_r)/(Nx_r)$ is above a minimum bound. So, for a given function $F(\cdot)$ and a given N, (2.a) holds for rates r that are above a lower bound; or, for a given value of r, when the number N of fishing units is below an upper bound. Thus, the result (2.a) holds for 'relatively high values' of r and/or 'relatively low values' of N. The result at (2.b) will hold for the opposite parametric value ranges.

Notice that the condition 'β small enough' in Proposition 4.4 implies the validity of one of the two conditions needed for ensuring the existence and uniqueness of a positive steady-state solution in this model (see Proposition 3.1). The second condition needed is $r < F'(0)$, which defines an upper bound on r such that the stationary solution $x(\beta) > 0$. Given these two conditions, the results in Part 2 of Proposition 4.4 then define value ranges for N, given a function $F(\cdot)$, which determine whether the result (2.a) or (2.b) holds.

Indeed, considering $r \in (0, F'(0))$ and a function $F(\cdot)$ satisfying the properties defined at Section 2, the result (2.b) will hold for $N/e > F'(0)$ (which imposes a lower bound on N); whereas the result (2.a) will hold for $N/e < F(x')/x'$ (i.e., an upper bound on N), where x' is the element such that $F(x') = \max\{F(x) : 0 < x < K\}$ and K is the saturation constant defined at Section 2. While for intermediate N values such that $F(x')/x' < N/e < F'(0)$, the result (2.b) will then hold for sufficiently small values of r (near to zero), whereas the result (2.a) will hold for higher values of r (indeed, near to $F'(0)$).

Recall that a higher r implies a greater opportunity cost of (investing in) keeping an additional unit of x at sea, which in turn implies, *ceteris paribus*, a social planner's lower demand valuation for investing in x. Thus, for higher values of r, the stationary solution for x should be at a lower level, while the stationary optimal fishing effort u should be at a higher level (keeping constant all other factors). Remember that the social planner's valuation of marginal investments in x is the value of the co-state variable λ. A similar 'downwardly effect', on the social planner's valuation of marginal investments in x, will be associated to lower values of N (again keeping constant all other factors).

Therefore, the result (2.a) will hold for (N, r) pairs such that the social planner's valuation of marginal investments in x is relatively low (i.e., $\lambda < p - c$). While the result (2.b) will hold for (N, r) pairs such that the social planner's valuation of marginal investments in x is relatively higher ($\lambda > p - c$); the latter case, due to a relatively larger N or a sufficiently lower r.

If a fishery were to face an exogenously-driven change in its β parameter, it could be of interest to know ex-ante not only how the equilibrium shadow price of the stock x would change, but also how other key variables in the fishery would respond. Accordingly, in what follows we analyse the equilibrium responses, to changes in parameter β, in the fishery's stationary catch, stationary fishing effort and the corresponding input's marginal productivities.

Firstly, in the following proposition, relation $\frac{dh}{d\beta} > 0$ is proven at the equilibrium for sufficiently small values of β:

[5]This term can be interpreted as the biological rate of return (per fishing unit) from marginal investment in x, when $x = x_r$.

Proposition 4.5. *For β positive near to zero, the equilibrium harvesting $h(\beta)$ decreases when β decreases.*

Proof. See Appendix A.4. $\square$

Secondly, we also know by the result in Part 1 of Proposition 4.4 that, at the stationary equilibrium, $\frac{dx}{d\beta} > 0$ for β positive near to zero. Thirdly, and again for β positive small enough, it can be proved that, at the stationary equilibrium, $\frac{d}{d\beta}\left(\frac{\partial h}{\partial x}\right) > 0$, as we establish in the next proposition.

Proposition 4.6. *For β positive near to zero, the marginal equilibrium harvesting $\frac{\partial h}{\partial x}$ decreases when β decreases.*

Proof. See Appendix A.5. $\square$

Therefore, for considered β, we know that a reduction in β will always reduce the steady state levels of x and h. So, how can it then be that, for β near to zero, the sign of $\frac{d\lambda}{d\beta}$ changes as a function of a critical value for the parametric condition $F(x_r)/(Nx_r) > 0$? The answer lies in the marginal effects of changes in β upon the stationary value of the fishing effort u. The following Propositions and Corollaries provide the answer.

Proposition 4.7. *The marginal harvesting productivity of u, at equilibrium, has the same monotonicity properties that the shadow price when β varies. Indeed,*

$$\frac{d}{d\beta}\left(\frac{\partial h}{\partial u}(x(\beta),\lambda(\beta))\right) = \left(\frac{Nc}{(p-\lambda(\beta))^2}\right)\frac{d\lambda(\beta)}{d\beta}. \tag{4.6}$$

Proof. See Appendix A.6. $\square$

Corollary 4.8. *For β positive near to zero, we have that:*

1. *if* $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 > 0$ *then* $\frac{d}{d\beta}\left(\frac{\partial h}{\partial u}\right) < 0$.

2. *if* $\ln\left(\frac{F(x_r)}{Nx_r}\right) + 1 < 0$ *then* $\frac{d}{d\beta}\left(\frac{\partial h}{\partial u}\right) > 0$.

Proof. This is a direct consequence of Propositions 4.4 and 4.7. $\square$

Naturally, we are interested in the monotonicity properties described in Parts 1 and 2 of Collorary 4.8 for the case when $p > c$, which is a necessary condition to obtain stationary solutions of economic interest, i.e. where $x(\beta)$, $\lambda(\beta)$ and $u(\beta)$ are all strictly positive (i.e., with $\lambda < p$). In this setting (β near to zero), the (N, r) pairs that guarantee the validity of the result (2.a) in Proposition 4.4 also imply that the marginal productivity of fishing effort u increases as β declines, all other factors remaining constant; while the opposite effect on the marginal productivity of u prevails for (N, r) pairs such that the result (2.b) in Proposition 4.4 is valid. Thus, for initial λ values such that $\lambda < p - c$, which are compatible with relatively higher values of r and/or relatively lower values of N, the result in Part 1 of Corollary 4.8 will prevail; and *vice versa* for the result in Part 2 of this Corollary.

The next Proposition 4.9 and its corresponding Corollary 4.10 describe the marginal effect on the stationary solution $u(\beta)$ as the value of β changes.

Proposition 4.9. *The limit of $\frac{du}{d\beta}$ when β goes to zero is:*

$$\lim_{\beta\to 0}\frac{du}{d\beta}=\frac{F(x_r)}{N}\left(\ln\left(\frac{F(x_r)}{Nx_r}\right)-\frac{rc}{F''(x_r)(p-c)x_r}\right).$$

Proof. See Appendix A.7. □

Corollary 4.10. *For β positive near to zero, if $\frac{F(x_r)}{Nx_r}>1$, then $\frac{du}{d\beta}>0$ and $\frac{d\lambda}{d\beta}<0$.*

Proof. The sign of $\frac{du}{d\beta}$ is a direct consequence of Proposition 4.9. Indeed, the strict concavity of F implies that $\frac{F(x_r)}{N}\left(\ln\left(\frac{F(x_r)}{Nx_r}\right)-\frac{rc}{F''(x_r)(p-c)x_r}\right)>0$ whenever $\frac{F(x_r)}{Nx_r}>1$. The sign of $\frac{d\lambda}{d\beta}$ is trivially obtained from Proposition 4.4 because $\frac{F(x_r)}{Nx_r}>1$ implies that $\ln\left(\frac{F(x_r)}{Nx_r}\right)+1>0$. □

To examine the implications of the last two results, let us consider some specific parametric configurations. Firstly, consider intermediate values of N such that $F(x')/x'<N/e<F'(0)$, where x' is the element such that $F(x')=\max\{F(x):0<x<K\}$ and $K>0$ is the saturation constant. In this setting, we know that the result (2.b) in Proposition 4.4 will prevail for r sufficiently small (close to zero). Under these conditions, the result in Proposition 4.9 implies $\frac{du}{d\beta}<0$. Recall as well that for any (N,r) pair such that the result (2.b) in Proposition 4.4 is valid, the marginal productivity of u will increase as β declines. Therefore, for (N,r) pairs such that the social planner's λ value is greater than $p-c>0$, a reduction in β will imply that the stationary solutions for λ and x will both decline, whereas the stationary fishing effort will increase.

Secondly, let us keep the focus on any (N,r) pair such that the result (2.b) in Proposition 4.4 is valid, and suppose that the harvesting business is very profitable, in the sense that $p-c$ is large enough. In this case, the result in Proposition 4.9 will again imply $\frac{du}{d\beta}<0$. Thus, as long as the harvesting activity is very profitable, and the present value of the stream of future profits is sufficiently large (given a relatively low value of r and/or a relatively large value of N), the social planner's optimal reaction to a lower value of β would be to reduce its investment in x while also increasing the stationary fishing effort u.

Thirdly, let us now consider any given (N,r) pair such that all three stationary solutions $(x(\beta),\lambda(\beta),u(\beta))$ are strictly positive, and suppose that $(p-c)$ is positive but now small enough. In this case, the result in Proposition 4.9 implies $\frac{du}{d\beta}>0$, given the strict concavity of function $F(\cdot)$. Therefore, when β declines and there are still profits to be made from fishing x but the profit per unit of fishing effort is small enough, the social planner's optimal policy will again be to invest in a lower stationary x, but now also reducing the stationary fishing effort level.

Finally, let us keep the focus on any (N,r) pair such that all three stationary solutions $(x(\beta),\lambda(\beta),u(\beta))$ are strictly positive. Suppose now, for any given value of $p>c$, that r has a 'relatively high' value, i.e. close enough to $F'(0)$, so that x_r gets close to zero. In this case, the Proposition 4.9 implies $\frac{du}{d\beta}>0$. Thus, a lower value of β will induce a social planner's lower investment in stationary x as well as the choice of a lower stationary fishing effort u (recall that with r close to $F'(0)$, the planner's λ value will increase as β declines). In a consistent way, Corollary 4.10 states that $\frac{du}{d\beta} > 0$ will prevail for sufficiently high values of r and/or sufficiently low values of N, such that $F(x_r)/Nx_r > 1$.

In Table 1 we provide a summary of the main derived results.

Parametric situation	$\frac{dx}{d\beta}$	$\frac{du}{d\beta}$	$\frac{dh}{d\beta}$	$\frac{d}{d\beta}\left(\frac{\partial h}{\partial u}\right)$	$\frac{d}{d\beta}\left(\frac{\partial h}{\partial x}\right)$
$F(x_r)/(Nx_r) > 1/e$	+	?	+	—	+
$F(x_r)/(Nx_r) < 1/e$	+	?	+	+	+
$F(x_r)/(Nx_r) > 1$	+	+	+	—	+

Table 1: Summary of dependencies with respect to β: increasing (+) and decreasing (−).

5 Final Remarks

Empirical evidence on small pelagic fish stocks confirms the relevance of analyzing fishery contexts with β positive but small enough. For these fish species, special emphasis should be put on the effects of relatively stock-insensitive, or even declining, average harvesting costs at reduced stock levels. Consistently, we have therefore focused on cases with β positive that may even tend to zero, subject to $\alpha + \beta = 1$.

Our modeling focus has been to study the sensitivity of a fishery's stationary equilibrium with respect to changes in β and, from the assumption $\alpha + \beta = 1$, consequently with respect to changes in α. The effects on stock, catch and fishing effort levels are described as functions of relative values of biological growth, discount rates and the total number of fishing units, all of which affect the fish stock's scarcity value. To our knowledge, the analysis of a fishery's equilibrium response to changing values of β has not been sufficiently developed in the bio-economic mathematical economic modeling of commercial fisheries, which has normally assumed either linear (Schaefer type) harvest functions or, in very few cases, Cobb-Douglas harvest functions with fixed (constant) values for β and α.

We believe it is important to develop further results and implications about the possible effects from changing values of β on a fishery's equilibrium and its properties, specially for cases with $0 \leq \beta < 1$. Further analysis on this issue is not only justified because of its particular relevance for small pelagic stocks, but as it can also help enlightening a range of different policy issues (e.g., avoidance of fishing collapse; and regulatory choices between using landing fees or harvest quotas) whose solution sets are conditioned by β-related effects.

We have studied a single species fishery equilibrium responses to changing values of β in the context of a highly stylized, textbook-like, one-dimensional fish stock model. Its straightforward simplicity allows us to analyze the basic structure of these stationary equilibrium responses while also avoiding getting results whose intuitive reading may be obscured by unenlightening technical details. Our analysis has focused on the case of $\alpha + \beta = 1$ for two main reasons. First, it is consistent with the empirical feature of relatively stock-insensitive or even declining average harvesting costs at reduced stock levels. And second, because of its simplicity and mathematical tractability.

Notice that the incentives for a more intensive exploitation of x as β declines become even stronger under the assumption of $\alpha + \beta = 1$, relative to the case of declining β (tending to zero) and a given (fixed) α (also < 1). However, the latter case should only imply quantitative differences with respect to our analysis; qualitative aspects, such as the type of equilibrium and its convergence properties, would remain unaffected.

The social planner assumption is meant to imply that the harvesting problem is defined independently from the ownership of the natural resource. That is why we assume that the planner has full control over the harvesting of the resource. Thus, we ignore the analysis of strategic interactions that can emerge in non-cooperative harvesting games arising as the result of a common-pool resource.

To extend our line of discussion to fishery contexts with $\alpha \geq 1$ would imply interesting qualitative differences with respect to our case. This variant would introduce interesting complications such as multiple equlibria and their corresponding convergence trajectories and resulting stability properties.

A Appendix: Proofs

A.1 Proof of Proposition 3.1

For the sake of simplicity, we denote the steady states of (2.8) by $x^* = x^*(\beta)$ and $\lambda^* = \lambda^*(\beta)$.

In order to prove the existence of (x^*, λ^*), let us define the following function

$$g(x) = \frac{c}{\alpha}\left(\frac{F(x)}{Nx}\right)^{\frac{\beta}{\alpha}}\left(\beta\frac{F(x)}{x} + r - F'(x)\right) + F'(x)p.$$

Note that

$$g(x) \to \frac{c}{\alpha}\left(\frac{F'(0)}{N}\right)^{\frac{\beta}{\alpha}}(\beta F'(0) + r - F'(0)) + F'(0)p, \qquad \text{when } x \to 0.$$

This last limit is strictly greater than rp. On the other hand, $g(K) = F'(K)p < 0 < rp$. Therefore, there exists $x^* \in [0, K]$ such that $g(x^*) = rp$. Then, defining

$$\lambda^* = p - \frac{c}{\alpha}\left(\frac{F(x^*)}{Nx^*}\right)^{\frac{\beta}{\alpha}},$$

it follows that (x^*, λ^*) is a steady state of (2.8) and relation (3.1) is satisfied.

We now proceed to prove that $(x^*, \lambda^*) \in (x_r, +\infty) \times (\lambda_\beta, p)$. Note that, since F es strictly concave, $x = x_r$ is the only point satisfying $F'(x) = r$. Then, since $F(x_r) \neq 0$ (otherwise

$F(x) < 0$ for all $x < x_r$ close enough to x_r which contradicts the positivity of F on $[0, K]$), it necessarily holds that $\lambda < p$. Also, cases when $x^* = 0$ or $\lambda^* = 0$ are trivially discarded. So, (x^*, λ^*) is a steady state if and only if

$$\frac{F(x^*)}{Nx^*} = \phi^\alpha(\lambda^*) \tag{A.1}$$

$$F'(x^*) - r = \Phi(\lambda^*), \tag{A.2}$$

where the auxiliar function Φ is defined as follows

$$\Phi(\lambda) = \frac{\beta N \phi^\alpha(\lambda)(\lambda - p)}{\lambda}. \tag{A.3}$$

Notice that Φ has a minimum over $(-\infty, 0)$ at $\lambda_m = -\beta p/(1-\beta)$ (Indeed, $\Phi'(\lambda) < 0$ when $\lambda \in (-\infty, \lambda_m)$ and $\Phi'(\lambda) > 0$ when $\lambda \in (\lambda_m, 0)$). So, the hypotheses on $F'(0)$ and the strictly concavity of F imply that

$$F'(x^*) - r \leq F'(0) - r < F'(0) < N((1-\beta)p/c)^{\frac{1-\beta}{\beta}} < N(p/c)^{\frac{1-\beta}{\beta}} = \Phi(\lambda_m),$$

which together with (A.2) discards the case when $\lambda^* < 0$. Consequently, x^* should be necessarily strictly greater than x_r, because otherwise left and right terms in equality (A.2) have opposite signs.

We finally note that $\lambda > \bar{\lambda}_\beta$ follows from the relations:

$$\phi^\alpha(\bar{\lambda}_\beta) = \frac{F(\bar{x})}{N\bar{x}} > \frac{F(x^*)}{Nx^*} = \phi^\alpha(\lambda^*)$$

where the inequality is due to the monotonicity of function $x \to F(x)/x$, which is a consequence of the strict concavity of F.

We finish this proof by showing the uniqueness of the steady state (x^*, λ^*). Consider two steady states (x_1, λ_1) y (x_2, λ_2). Since functions $x \to F(x)/x$ and $\lambda \to \phi^\alpha(\lambda)$ are decreasing, we have the following equivalences:

$$x_1 \leq x_2 \Leftrightarrow \frac{F(x_2)}{Nx_2} \leq \frac{F(x_1)}{Nx_1} \Leftrightarrow \phi^\alpha(\lambda_2) \leq \phi^\alpha(\lambda_1) \Leftrightarrow \lambda_1 \leq \lambda_2.$$

On the other hand, it is easy to verify that function Φ is increasing on $(\bar{\lambda}_\beta, p)$. Consequently, we also have the equivalences:

$$\lambda_1 \leq \lambda_2 \Leftrightarrow \Phi(\lambda_1) \leq \Phi(\lambda_2) \Leftrightarrow F'(x_1) \leq F'(x_2) \Leftrightarrow x_2 \leq x_1.$$

We thus conclude that $x_1 = x_2$ and $\lambda_1 = \lambda_2$.

A.2 Proof of Proposition 4.1

For a given $\beta \in (0,1)$ such that there is only one equilibrium point (see Proposition 3.1), the Jacobian matrix of the right-hand-side function of (2.8), at the steady state $(x^*(\beta), \lambda^*(\beta))$, is given by

$$J(\beta) := J(x^*(\beta), \lambda^*(\beta)) = \begin{bmatrix} F'(x^*(\beta)) - \frac{F(x^*(\beta))}{x^*(\beta)} & \frac{\alpha F(x^*(\beta))}{\beta(p-\lambda^*(\beta))} \\ \\ -\lambda^*(\beta)F''(x^*(\beta)) & r - F'(x^*(\beta)) + \frac{F(x^*(\beta))}{x^*(\beta)} \end{bmatrix}. \tag{A.4}$$

The determinant of $J(\beta)$ is then computed as follows

$$\det(J(\beta)) = A(\beta) - \frac{B(\beta)}{\beta}, \tag{A.5}$$

where

$$A(\beta) \;:=\; \left(F'(x^*(\beta)) - \frac{F(x^*(\beta))}{x^*(\beta)}\right)\left(r - \left(F'(x^*(\beta)) - \frac{F(x^*(\beta))}{x^*(\beta)}\right)\right), \tag{A.6}$$

$$B(\beta) \;:=\; -\lambda^*(\beta)F''(x(^*\beta))\frac{\alpha F(x^*(\beta))}{(p-\lambda^*(\beta))}. \tag{A.7}$$

So, since F is strictly concave and $\lambda(\beta) > 0$, it follows that $A(\beta) < 0$ and $B(\beta) > 0$. We thus obtain that $\det(J(\beta)) < 0$, for all β such that assumptions of Proposition 3.1 hold true. Hence, the implicit function theorem implies the existence of two continuously differentiable mappings of β, simply denoted here by $x(\cdot)$ and $\lambda(\cdot)$ (the range sets of these functions are obtained in Proposition 3.1), satisfying (4.1).

Since $x(\beta)$ and $\lambda(\beta)$ remain in the compact set $C = [x_r, K] \times [0, p]$, in order to prove the convergences of $x(\beta)$ and $\lambda(\beta)$, we only need to prove that any converging subsequence has x_r and $p - c$, respectively, as their limit points. Consider then any sequence β_k converging to zero, when $k \to +\infty$, such that $x(\beta_k) \to \tilde{x}$ and $\lambda(\beta_k) \to \tilde{\lambda}$ for some $\tilde{x}$ and $\tilde{\lambda}$ in C. Since $\bar{\lambda}_\beta \to p - c$ when $\beta \to 0$ (see Proposition 4.1), we can ensure that $\tilde{\lambda} \geq p - c > 0$. Moreover, the first equation in (4.1) gives us the following relation:

$$\lambda(\beta) = p - \left(\frac{c}{1-\beta}\right)\left(\frac{F(x(\beta))}{Nx(\beta)}\right)^{\frac{\beta}{1-\beta}},$$

which implies that $\tilde{\lambda} = p - c$ provided that $F(\tilde{x}) \neq 0$.

Let us prove this claim. We argue by contradiction. Suppose that $F(\tilde{x}) = 0$, then we obtain from (4.1) that $\tilde{\lambda}(F'(\tilde{x}) - r) = 0$, and consequently $F'(\tilde{x}) = r$. This holds only if $\tilde{x} = x_r$ (because, since F is strictly concave, $x = x_r$ is the only point satisfying $F'(x) = r$). However, this contradicts the fact that $F(x_r) \neq 0$ (otherwise $F(x) < 0$ for all $x < x_r$ close enough to x_r which contradicts the positivity of F on $[0, K]$). Hence $\tilde{\lambda} = p - c$. The second equation in (4.1) allows us to conclude that $\tilde{x} = x_r$. The desired convergences of $x(\beta)$ and $\lambda(\beta)$ are thus established.

Finally, the convergence of $u(\beta)$ follows directly from equality (4.3).

A.3 Proof of Proposition 4.4

From the implicit function theorem, the derivative of $x(\beta)$ can be computed as follows

$$\begin{aligned}\frac{dx(\beta)}{d\beta} = \frac{N}{\beta \det J(\beta)} \Bigg[& \frac{-F(x(\beta))}{Nx(\beta)} \left(\frac{1}{1-\beta} \ln \left(\frac{F(x(\beta))}{Nx(\beta)} \right) + 1 \right) \{ rx(\beta) - x(\beta)F'(x(\beta)) + \beta F(x(\beta)) \} \\ & - (1-\beta) \frac{F(x(\beta))^2}{Nx(\beta)} \Bigg],\end{aligned}$$

where $J(\beta) := J(x(\beta), \lambda(\beta))$ is the Jacobian matrix of the RHS of (2.8) at the steady state $(x(\beta), \lambda(\beta))$. On the other hand, equation (A.5) established that $\det J(\beta) = A(\beta) - B(\beta)/\beta < 0$, for all $\beta \in (0,1)$, where $A(\beta)$ and $B(\beta)$ were described in (A.6) and (A.7), respectively. By noting that

$$\begin{aligned}\lim_{\beta \to 0} A(\beta) &= \left(r - \frac{F(x_r)}{x_r} \right) \frac{F(x_r)}{x_r} \\ \lim_{\beta \to 0} B(\beta) &= -(p-c)F''(x_r)\frac{F(x_r)}{c},\end{aligned}$$

we conclude $\beta \det J(\beta) \to (p-c)F''(x_r)\frac{F(x_r)}{c}$, when $\beta \to 0$. This limit value is negative because of the strict concavity of F. Therefore,

$$\lim_{\beta \to 0} \frac{dx(\beta)}{d\beta} = -\frac{cF(x_r)}{F''(x_r)(p-c)x_r} > 0.$$

Analogously, from the implicit function theorem, the derivative of $\lambda(\beta)$ can be computed as follows

$$\begin{aligned}\frac{d\lambda(\beta)}{d\beta} =& \frac{N}{\beta \det J(\beta)} \Bigg[\frac{-F(x(\beta))}{Nx(\beta)} \left(\frac{1}{1-\beta} \ln \left(\frac{F(x(\beta))}{Nx(\beta)} \right) + 1 \right) \{ \lambda(\beta)F''(x(\beta))x(\beta) \\ & + \beta \left(F'(x(\beta)) - \frac{F(x(\beta))}{x(\beta)} \right) (p - \lambda(\beta)) \} + \beta \left(F'(x(\beta)) - \frac{F(x(\beta))}{x(\beta)} \right) (p - \lambda(\beta)) \frac{F(x(\beta))}{Nx(\beta)} \Bigg].\end{aligned}$$

Hence

$$\lim_{\beta \to 0} \frac{d\lambda(\beta)}{d\beta} = -c \left[\ln \left(\frac{F(x_r)}{Nx_r} \right) + 1 \right].$$

We have thus concluded (4.4) and (4.5).

Finally, from (4.4) and (4.5), we deduce that $\frac{dx(\beta)}{d\beta} > 0$ and that $\frac{d\lambda(\beta)}{d\beta}$ has the opposite sign of $\ln \left(\frac{F(x_r)}{Nx_r} \right) + 1$ when β is small enough. The proposition follows.

A.4 Proof of Proposition 4.5

Propositions 4.1 and 4.4 imply that $x(\beta)$ decreases to x_r when $\beta \to 0$. This in particular implies that $F'(x(\beta)) > 0$ and $\frac{dx(\beta)}{d\beta} > 0$ when β is small enough. Hence, we obtain from (4.3) that

$$\frac{\partial h(x(\beta), u(\beta))}{\partial \beta} = F'(x(\beta))\frac{dx(\beta)}{d\beta} > 0, \quad \text{for all } \beta \text{ small enough.}$$

A.5 Proof of Proposition 4.6

The partial derivative of h with respect to x is given by

$$\frac{\partial h}{\partial x}(x,u) = N\beta\left(\frac{u}{x}\right)^{1-\beta}.$$

Therefore, at the equilibrium $(x(\beta), u(\beta))$, it holds that

$$\frac{\partial}{\partial \beta}\left(\frac{\partial h}{\partial x}(x(\beta), u(\beta))\right) = N\left(\frac{u(\beta)}{x(\beta)}\right)^{1-\beta}\left(1 - \beta \ln\left(\frac{u(\beta)}{x(\beta)}\right)\right).$$

However, it follows from (4.2) that $\beta \ln\left(\frac{u(\beta)}{x(\beta)}\right) = \frac{\beta}{\alpha}\ln\left(\frac{F(x(\beta))}{Nx(\beta)}\right)$, which tends to 0 when $\beta \to 0$. We thus conclude that $\frac{\partial}{\partial \beta}\left(\frac{\partial h}{\partial x}(x(\beta), u(\beta))\right)$ is positive when β is small enough.

A.6 Proof of Proposition 4.7

The partial derivative of h with respect to u is given by

$$\frac{\partial h}{\partial u}(x,u) = \alpha N\left(\frac{x}{u}\right)^{\beta}.$$

Then, at the equilibrium $(x(\beta), u(\beta))$, we obtain from (4.2) the expression

$$\frac{\partial h}{\partial u}(x(\beta), u(\beta)) = \frac{Nc}{(p - \lambda(\beta))},$$

and relation (4.6) is obtained by deriving the above equality with respect to β.

A.7 Proof of Proposition 4.9

Define the function $\zeta(\beta, x, u) = F(x) - Nu^{(1-\beta)}x^{\beta}$. From (4.3) we obtain that

$$\zeta(\beta, x(\beta), u(\beta)) = 0 \qquad \forall\ \beta \in (0,1).$$

Deriving this equality with respect to β we obtain

$$\partial_\beta \zeta + \partial_x \zeta\ \frac{dx}{d\beta} + \partial_u \zeta\ \frac{du}{d\beta} = 0. \tag{A.8}$$

It is straightforward to check that the partial derivatives of the function ζ are given by:

$$\begin{aligned}
\partial_\beta \zeta &= \frac{Nu}{\alpha}\left(\frac{x}{u}\right)^{\beta} \ln\left(\frac{F(x)}{Nx}\right) \\
\partial_x \zeta &= F'(x) - \beta \frac{F(x)}{x} \\
\partial_u \zeta &= -N\alpha \left(\frac{x}{u}\right)^{\beta}.
\end{aligned}$$

So, Proposition 4.1 implies that $\partial_\beta \zeta \to F(x_r)\ln\left(\frac{F(x_r)}{Nx_r}\right)$, $\partial_x \zeta \to r$, $\partial_u \zeta \to -N$ when $\beta \to 0$. These limits, expression (4.4) for the limit of $\frac{dx}{d\beta}$ when $\beta \to 0$, and (A.8) give us the desired result.

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